REPORT No. 882

FREQUENCY-RESPONSE METHOD FOR DETERMINATION OF DYNAMIC STABILITY CHARACTERISTICS OF AIRPLANES WITH AUTOMATIC CONTROLS

By Harry Greenberg

SUMMARY

A frequency-response method for determining the critical control-gearing and hunting oscillations of airplanes with automatic pilots is presented. The method is graphical and has several advantages over the standard numerical procedure based on Routh's discriminant. The chief advantage of the method is that direct use can be made of the measured response characteristics of the automatic pilot. This feature is especially useful in determining the existence, amplitude, and frequency of the hunting oscillations that may be present when the automatic pilot has nonlinear dynamic characteristics.

Several examples are worked out to illustrate the application of the frequency-response method in determining the effect of automatic-pilot lag or lead on critical control gearing and in determining the amplitude and frequency of hunting. It is shown that the method may be applied to the case of a control geared to airplane motions about two axes.

INTRODUCTION

The increased use of automatic control on aircraft (especially on pilotless aircraft) has focused attention on the proper design of the aircraft and associated control systems with a view toward obtaining satisfactory dynamic stability. The control system (autopilot) consists of the gyro, phase-shifting device, and servomotor. Factors in the control system that determine stability characteristics are control gearing, lag in the servomotor, and lead in the phase-shifting device. The stability characteristics of the airplane depend on airplane configuration and mass distribution.

The purpose of the present report is to give a method for analyzing the dynamic stability of an airplane with automatic control which makes direct use of the observed dynamic characteristics of the automatic pilot (herein called autopilot). The method separates the characteristics of the autopilot from those of the airplane. It, therefore, easily reveals the effects of modifications to the autopilot such as adding lead. The procedure is largely graphical. The method is similar in certain respects to that of Nyquist (reference 1), which was devised for electronic circuits but which has also been applied by some workers to the design of servomotors.

The frequency-response method was previously applied by Jones (reference 2) to calculate the hunting produced in airplanes with "flicker," that is, on-off control. If the servomotor (herein called servo) has a linear lag characteristic (lag independent of amplitude), then there will usually be an upper limit to the control gearing above which the airplane will be unstable. The frequency-response method determines this critical control gearing and the corresponding frequency for any type of lag or lead in the autopilot. It also indicates the changes to be made in the servo which will improve the stability of the airplane.

The utility of the frequency-response method is most apparent in the case of an actual servo with nonlinear lag characteristics. This method allows the use of measured characteristics of the servo that might be inconvenient to represent by a mathematical formula.

TERMINOLOGY AND SYMBOLS

The word "lead" is used in this report in two ways. It is the phase angle of control-surface deflection δ ahead of airplane deflection, say angle of pitch θ, and it is also used to indicate a device that causes an increase in the phase angle of lead of δ ahead of θ. These phase-shifting devices are herein called first-derivative lead and second-derivative lead.

The expression "linear autopilot" is used to indicate a servo that is acted upon by a force proportional to the input signal (airplane deflection) and resisted by a force proportional to the displacement and velocity of the output (control-surface motion).

The function of the automatic pilot is to apply a corrective control deflection in response to any deflection of the airplane. The "control gearing" is the ratio of the applied control deflection to the airplane deflection for very slow deflections (static condition). There may exist a "critical control gearing," which in nearly all practical cases is an upper limit beyond which dynamic instability occurs.

The following symbols are used:

a — ratio of control deflection due to second derivative of airplane displacement to that due to airplane displacement

r — ratio of control deflection due to first derivative of airplane displacement to that due to airplane displacement

R amplitude of airplane oscillation produced by unit amplitude oscillation of control surface

δ_r amplitude of control-surface oscillation required to maintain unit amplitude of airplane oscillations $(1/R)$

δ_p amplitude of control-surface oscillation produced by autopilot in response to airplane oscillation, divided by amplitude of airplane oscillation

δ_{p_1} δ_p for unit control gearing

k_{cr} critical value of control gearing

k_1 spring stiffness factor of servo and control surface

k_2 factor relating airplane deflection and force on servo

k control gearing, ratio of δ to θ in static condition (k_2/k_1)

c viscous damping factor of servo and control surface

l viscous lag factor which depends on c and k_1 (c/k_1)

m mass of movable part of servo and control surface

t time, seconds

T period of oscillation, seconds

δ angle of control-surface deflection in direction to reduce angle of pitch of airplane

ϵ_p phase angle of lead of δ ahead of θ when oscillating airplane forces control surface to oscillate

ϵ_r phase angle of lead of δ ahead of θ when oscillating control surface forces airplane to oscillate

θ angle of pitch of airplane

Θ signal fed into servo

ϕ angle of bank of airplane

ψ angle of yaw of airplane

Φ gyro displacement produced by airplane deflection

ω angular frequency, radians per second $(2\pi\times\text{Frequency})$

ω_{cr} angular frequency of servo and airplane when $k=k_{cr}$, radians per second

ω_n natural angular frequency of servo, radians per second

τ angle of tilt of gyro axis from X-axis of airplane

D differential operator (d/dt)

Subscript:

max maximum

BASIC PRINCIPLES OF FREQUENCY-RESPONSE METHOD

The response of the airplane to a sinusoidal control motion may be measured in flight or may be computed from the equations of motion of the airplane. (See appendix A.) The airplane response is sinusoidal and of the same frequency as the control motion and has an amplitude and phase that depends on this frequency. If the control motion is given by

$$\delta=\sin \omega t$$

then the response in pitch, for example, may be expressed as

$$\theta=R \sin (\omega t-\epsilon_r)$$

where R and ϵ_r both depend on ω, and R is the ratio of the amplitude of θ to the amplitude of δ when the control surface is forcing the airplane to oscillate. Changing the amplitude of δ produces a proportionate change in the amplitude of θ but does not affect the phase angle ϵ_r. The reciprocal of R gives the amplitude of δ required to sustain unit amplitude oscillation in θ and is denoted by δ_r; that is, if

$$\delta=\delta_r \sin \omega t$$

then the response will be

$$\theta=\sin (\omega t-\epsilon_r)$$

A plot of δ_r and ϵ_r against ω is then made and is referred to as the frequency response of the airplane.

The frequency response of the autopilot is obtained by measuring or computing the response of the control surface to a steady oscillation of the airplane. The response may be calculated if the dynamic constants of the autopilot are known, but it is usually easier to measure the response by oscillating the autopilot in the laboratory. These calculations or measurements give the response to

$$\theta=\sin \omega t$$

and this response is expressed by

$$\delta=\delta_{p_1} \sin (\omega t+\epsilon_p)$$

where δ_{p_1} is the amplitude of δ for unit control gearing and ϵ_p is the phase angle of lead of δ ahead of θ when the airplane is forcing the control surface to oscillate.

If at some value of the angular frequency ω the condition

$$\epsilon_p=\epsilon_r$$

exists, then the airplane connected to the autopilot will oscillate continuously at this frequency at constant amplitude, provided the control gearing is made equal to δ_r/δ_{p_1} at this value of ω; that is, the critical control gearing is given by

$$k_{cr}=\frac{\delta_r}{\delta_{p_1}}$$

at ω where $\epsilon_r=\epsilon_p$. This condition of neutral stability is shown schematically in figure 1. An example of a solution for k_{cr} is shown in figure 2. In this case $\omega_{cr}=6.2$ and $k_{cr}=\frac{\delta_r}{\delta_{p_1}}$ at this value of ω; therefore $k_{cr}=\frac{0.33}{0.91}=0.36$.

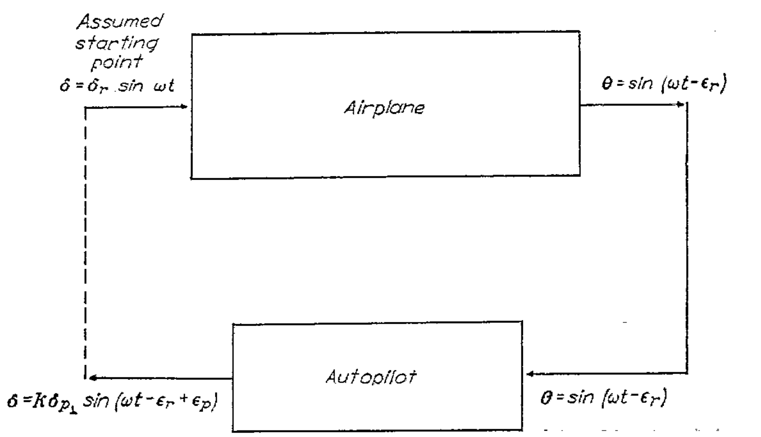

FIGURE 1.—Response of airplane to sinusoidal control motion and response of control to sinusoidal airplane motion. If $\epsilon_p=\epsilon_r$ and $k=k_{cr}=\frac{\delta_r}{\delta_{p_1}}$, the autopilot will sustain the oscillations.

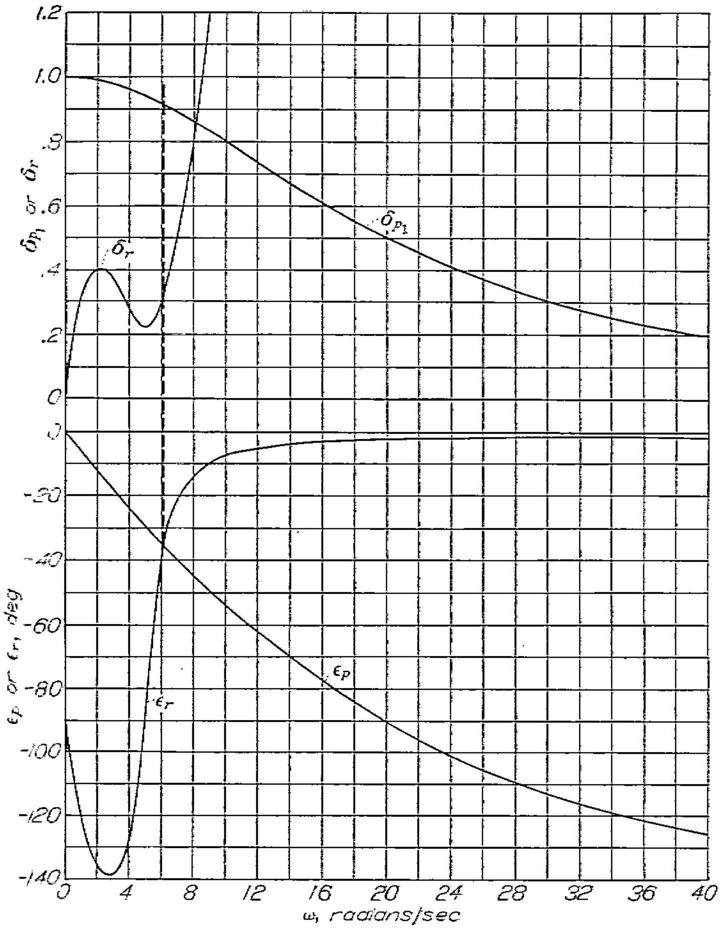

FIGURE 2.—Illustration of frequency-response method of determining critical control gearing. $\omega_{cr}=6.2$; $k_{cr}=0.36$. (Dotted line indicates where $\epsilon_p=\epsilon_r$.)

In almost all cases control gearings greater than k_{cr} result in instability and control gearings less than k_{cr} provide stability. Thus, k_{cr} is usually an upper limit to the permissible control gearing. More generally, the stable side of the boundary is determined by the relative slopes of the ϵ_r and ϵ_p curves at their intersection. If $\frac{d\epsilon_r}{d\omega}$ is greater than $\frac{d\epsilon_p}{d\omega}$, the system is stable below k_{cr} and vice versa. A proof of this rule is given in appendix B.

In the preparation of figure 2 and the examples that follow (figs. 3 to 8) a particular airplane configuration was assumed for the computation of δ_r and ϵ_r. (See appendix A for methods.) This hypothetical pilotless airplane has four fins 90° apart, a horizontal pair for pull-ups and a vertical pair for turns. The fins constitute the only lifting surfaces of the airplane and are equipped with ailerons connected to a roll gyro and servo that maintain the airplane in a position in which one pair of fins is always horizontal. The pitch and yaw control surfaces are trailing-edge flaps on the fins and are connected in pairs, one pair on the horizontal fins connected to a pitch gyro and servo and the other pair on the vertical fins connected to a yaw gyro and servo. It is assumed that the pitch and yaw motions do not interact with each other or with any accidental rolling motion that

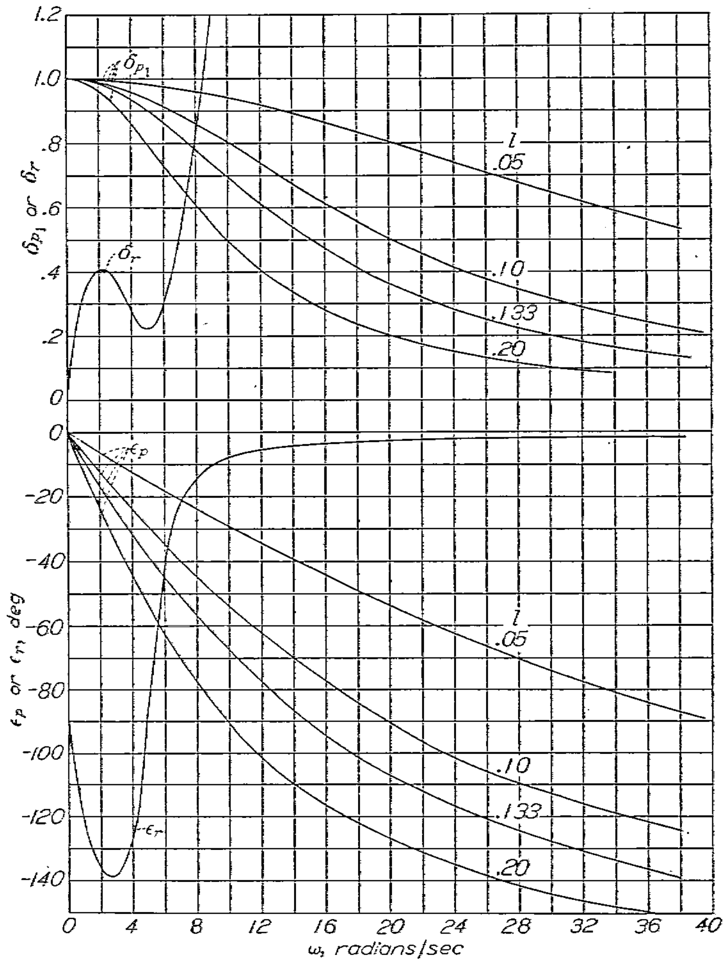

FIGURE 3.—Effect of lag in the servo on the frequency response of a critically damped linear servo. Natural frequency of servo $\omega_n=\frac{2}{l}$.

may exist. Figures 2 to 6 show the elevator-pitch stability. Rudder-yaw stability may be expected to be very nearly the same as elevator-pitch stability because of the configuration of the fins.

LINEAR AUTOPILOT

The motion of a control surface actuated by a simple linear autopilot is similar to that of a mass-spring-dashpot and may be represented by the equation

$$mD^2\delta+cD\delta+k_1\delta=k_2\theta \tag{1}$$

where each of the factors m, c, and k_1 is the sum of two parts, one part due to the servo and one part due to the control surface. For example, k_1 is the sum of the spring constant of the servo and the aerodynamic hinge-moment constant of the control surface. The term $k_2\theta$ represents a force proportional to the deviation in angle of pitch θ applied without lag.

Actually, the effect of control-cable flexibility is to introduce an additional degree of freedom into the system. This effect is probably small and may be taken into account either analytically or by flight measurements as mentioned in the section "Nonlinear Autopilot."

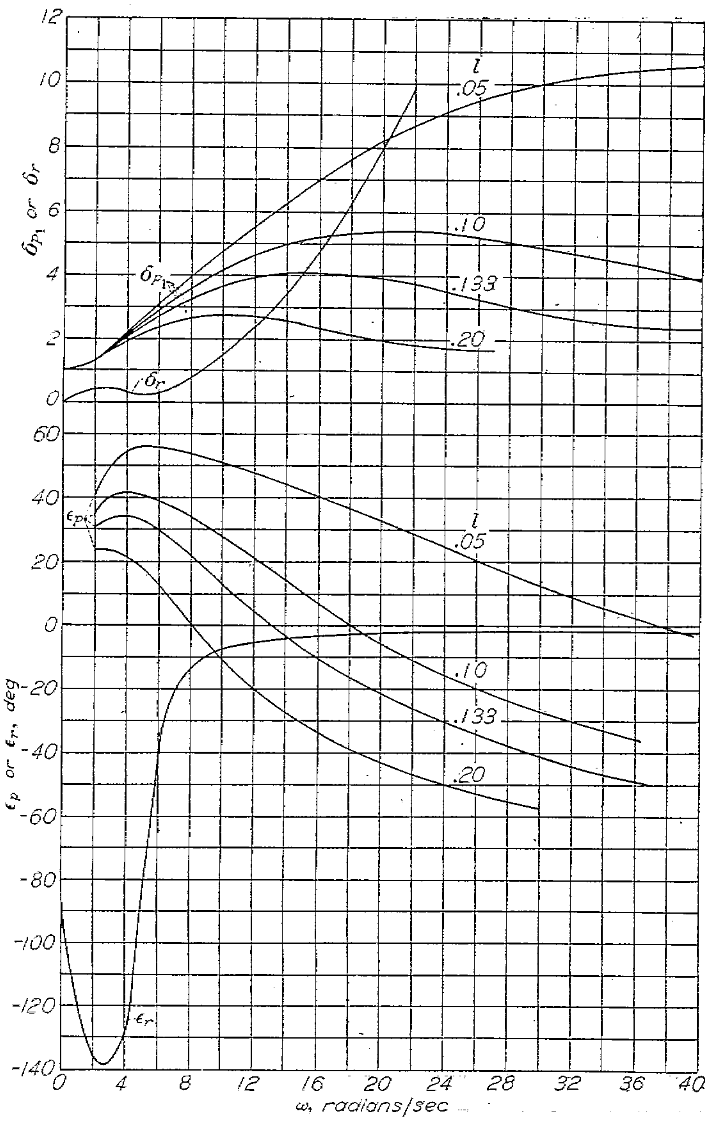

FIGURE 4.—Effect of first-derivative lead $\left(r=\frac{1}{2}\right)$ in the autopilot on the frequency response of a critically damped linear servo. Natural frequency of servo $\omega_n=\frac{2}{l}$.

Equation (1) may be expressed as

$$\delta\left(1+\frac{c}{k_1}D+\frac{m}{k_1}D^2\right)=\frac{k_2}{k_1}\theta \tag{2}$$

where k_2/k_1 is the control gearing k. The natural frequency of the servo is given by

$$\omega_n^2=\frac{k_1}{m}$$

and the damping depends on the quantity $l=\frac{c}{k_1}$. With these substitutions, equation (2) becomes

$$\delta\left(1+lD+\frac{D^2}{\omega_n^2}\right)=k\theta$$

The term lD represents viscous damping and the term D^2/ω_n^2 represents inertia reaction.

When the autopilot is oscillated in pitch, the control response δ lags behind θ. The lag usually decreases as the lag factor l decreases and as the natural frequency ω_n

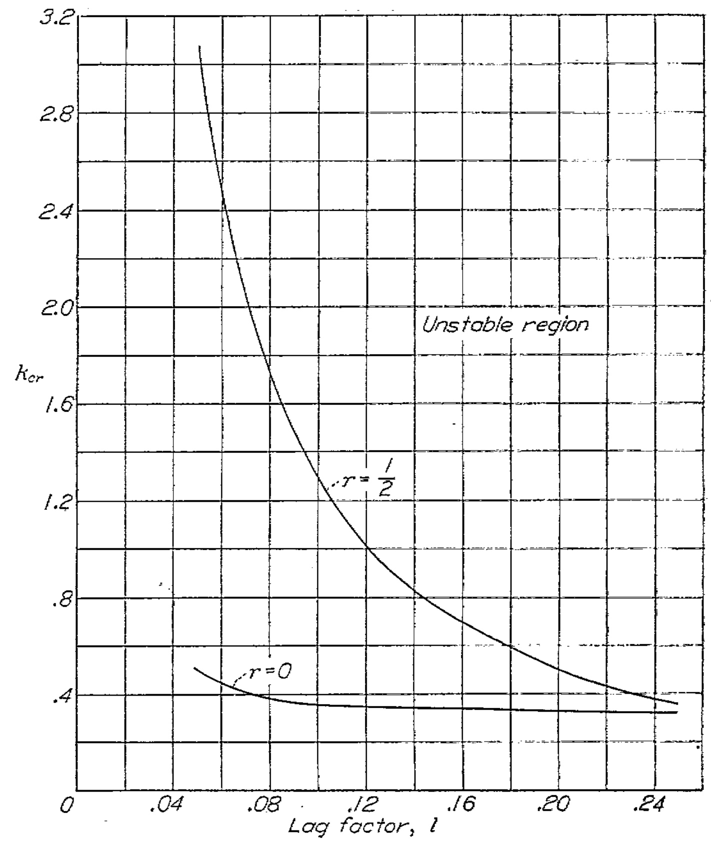

FIGURE 5.—Effect of lag and lead on the critical control gearing with a critically damped linear servo. Airplane characteristics of figure 2. $\delta=\frac{1+rD}{1+\frac{2}{\omega_n}D+\frac{D^2}{\omega_n^2}}$; $l=\frac{2}{\omega_n}$.

increases. It is convenient to choose ω_n equal to $2/l$, a value that gives critical damping. With this choice, the phase angle ϵ_p and the relative amplitude for unit control gearing δ_{p_1} of the response are presented in figure 3 for four values of l. Values of δ_r and ϵ_r for the assumed airplane are also given so that k_{cr} can be obtained for each value of l.

If first-derivative lead is added to the system, the equation of motion relating δ and θ becomes

$$\delta\left(1+lD+\frac{D^2}{\omega_n^2}\right)=k(1+rD)\theta$$

Figure 4 shows δ_{p_1} and ϵ_p for the same values of l and ω_n as figure 3, but with $r=\frac{1}{2}$. Comparison shows that the first-derivative lead produces an angle of lead that approaches 90° at high values of the angular frequency ω. This amount of lead is enough to override the lag for moderate values of l at low frequencies but not at high frequencies.

Values of k_{cr} obtained from figures 3 and 4 are shown in figure 5 plotted against the viscous lag factor l. Considerable increase in the stable range of control gearing is evident for small values of l with the first-derivative lead. Note how this improvement in stability diminishes at large values of l.

If both first-derivative lead and second-derivative lead are added, the response of the control is given by the equation

$$\delta=k\frac{1+rD+aD^2}{1+lD+\frac{D^2}{\omega_n^2}}\theta$$

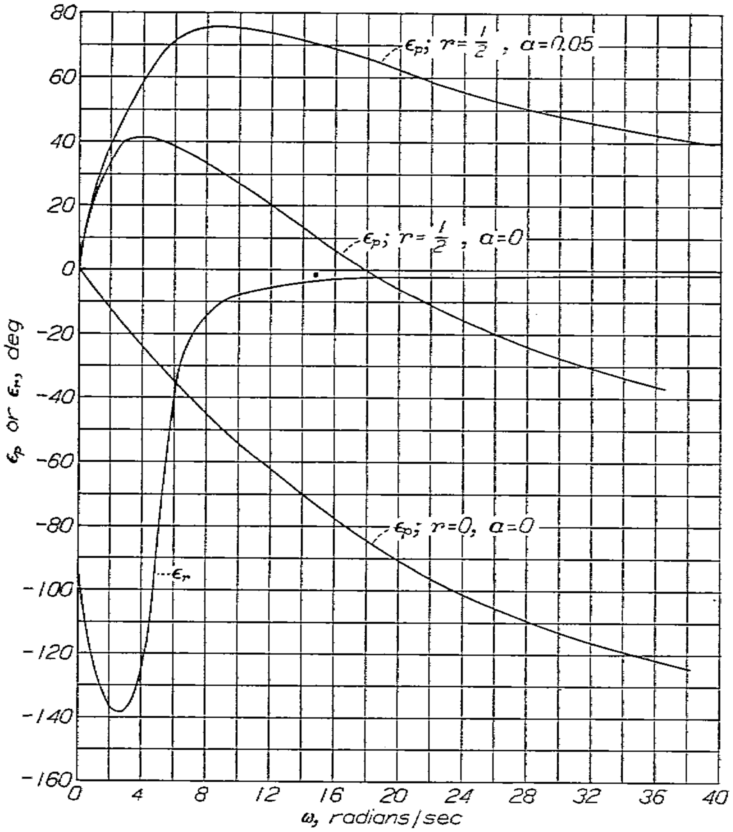

FIGURE 6.—Effect of first-derivative lead and second-derivative lead on the lag of the servo as a function of frequency. $\delta=\frac{1+rD+aD^2}{1+0.1D+0.0025D^2}\theta$.

It can be shown that if

$$r>l$$

and

$$a\omega_n^2>\frac{r}{l} \tag{3}$$

lead will exist at all frequencies. In this situation the airplane cannot oscillate regardless of the value of k or, stated differently, the critical control gearing is infinite. Figure 6 shows the resultant lag or lead for a critically damped servo for which $\omega_n=20$ radians per second and for various values of r and a. The top curve is for values of r and a that satisfy expression (3) and, therefore, result in lead over the entire frequency range.

NONLINEAR AUTOPILOT

Even if the servo is constructed to give a response proportional to a steady disturbance, its response to an oscillating disturbance is not proportional in practice to the amplitude of the disturbance because of the nonlinear dynamic characteristics of the servo. The lag is then a function of both amplitude and frequency. By finding experimentally the frequency response of the autopilot for a number of amplitudes and for a given control gearing, it is possible to determine the amplitude and frequency at which the airplane will oscillate when coupled to the autopilot at that particular value of control gearing. The condition for steady oscillations (hunting) is that $\delta_p=\delta_r$ and $\epsilon_p=\epsilon_r$ at some frequency. The frequency and amplitude of the steady oscillations of the airplane are the values of ω and θ_{max} at which $\delta_p=\delta_r$ and $\epsilon_p=\epsilon_r$.

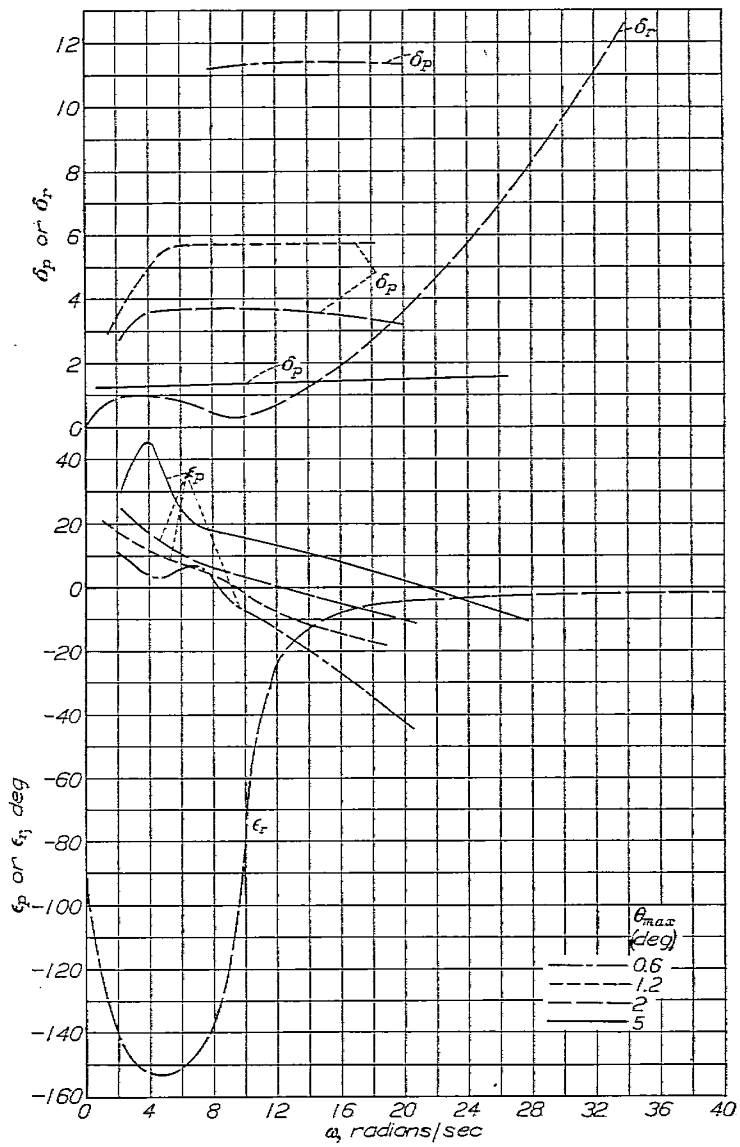

FIGURE 7.—Effect of amplitude on the frequency response of a servo, as measured on an oscillating table.

The stability of these constant-amplitude oscillations has a slightly different significance from that of linear systems. In linear systems the oscillations are, in general, either damped or undamped regardless of the amplitude. In the case of the hunting that may exist with a nonlinear servo the oscillations are said to be stable if, after a disturbance from their steady value, they tend to return to that steady value. The criterion for the stability of the steady oscillations follows from that for the linear autopilot and is

$$\frac{d(\epsilon_r-\epsilon_p)}{d\omega}\,\frac{d\,\frac{\delta_p}{\delta_r}}{d\theta_{max}}<0$$

The solution obtained is not exact because the autopilot response to sinusoidal airplane motion is not a pure sine wave but is distorted; however, the existence and approximate amplitude and frequency of hunting oscillations of the fundamental can be determined by the frequency-response method. The values of δ_p and ϵ_p used when the response is not sinusoidal are the values of the equivalent sinusoidal reponse which

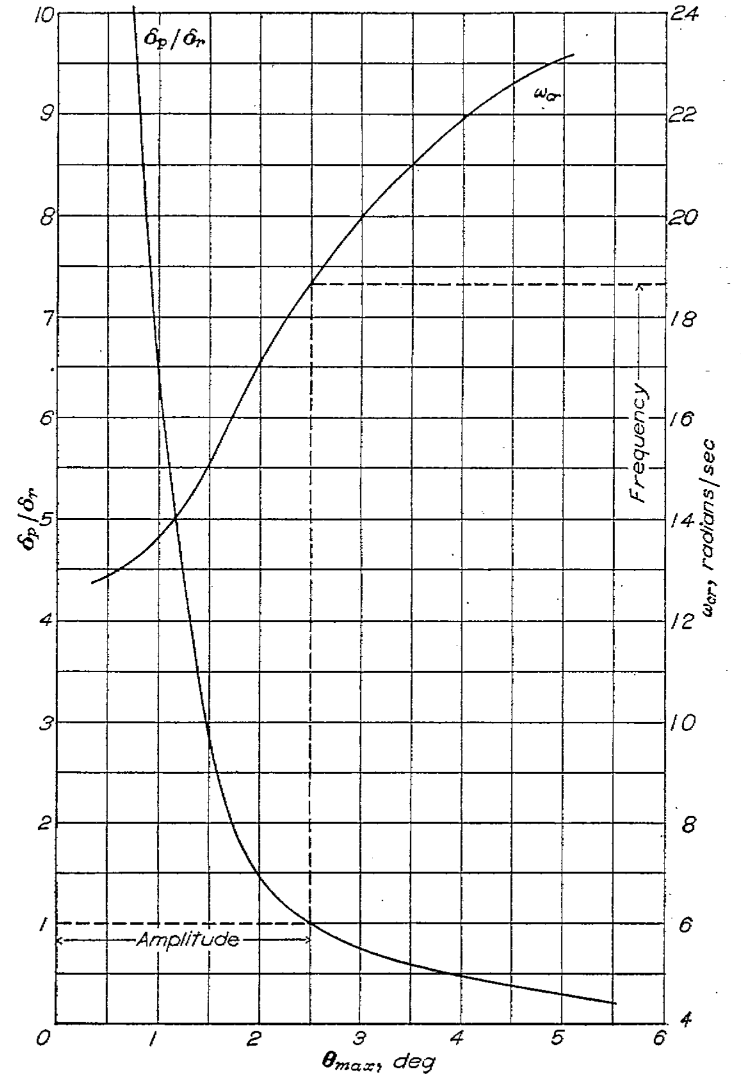

FIGURE 8.—Plot of δ_p/δ_r and ω_{cr} against θ_{max} obtained from figure 7, showing the amplitude and frequency of the hunting.

supplies the same energy per cycle and the same impulse per half cycle as the observed response. This condition leads to the following expressions for ϵ_p and δ_p:

$$\delta_p=\frac{\sqrt{A^2+B^2}}{\theta_{max}}$$

$$\epsilon_p=\tan^{-1}\frac{B}{A}$$

where

$$A=\frac{\omega}{2}\int_0^{T/2}\delta\,dt$$

$$B=\frac{1}{\pi}\int_0^{T}\delta\,d\theta$$

The value of δ in these integrals is the measured control-surface deflection in response to an oscillating motion of the airplane represented by $\theta=\theta_{max}\sin\omega t$. It turns out that B is exactly equal to the cosine coefficient of the fundamental frequency component of the Fourier series approximation to δ; however, A is not in general exactly equal to the first sine coefficient of the Fourier series. Ordinarily the oscillation in pitch is performed in the laboratory by mounting the autopilot on a table that can be made to oscillate.

A more exact method of determining the autopilot characteristics, which includes the effects of control-system flexibility and hinge moment, is to measure the response of the control surface and airplane in flight to a sinusoidal input signal to the servo. In order to obtain the frequency response of the control system to sinusoidal input signal, it is necessary to divide the vector output δ by the vector sum of input signal due to θ and input signal applied directly. Thus, the dynamic characteristics of the airplane and control system can be obtained at the same time. If the aircraft is not available for flight tests, one can "synthesize" the complete servo-control-system response from ground measurements of the servo and estimates of control-system flexibility, inertia, and aerodynamic hinge moments.

A typical application of the method for determining amplitude and frequency of hunting of an aircraft and autopilot with stops in the control system is shown in figures 7 and 8. The airplane used in this example is slightly different from that considered previously and, therefore, the δ_r and ϵ_r curves are slightly different from those in figures 2 to 6. Figure 7 contains the phase and amplitude of the experimental response of the autopilot plotted against frequency for a series of amplitudes of the input and also the corresponding calculated curves for the aircraft. For each amplitude the values of δ_p and δ_r are obtained at the frequency where $\epsilon_p=\epsilon_r$. The ratio δ_p/δ_r and the value of ω_{cr} (where $\epsilon_p=\epsilon_r$) are plotted against amplitude of pitch θ_{max} in figure 8. The amplitude and frequency of hunting occur where $\frac{\delta_p}{\delta_r}=1$. In this example the amplitude of the hunting is about 2.5° and the frequency is 18.7 radians per second.

Nonlinearity in the autopilot usually causes an increase of lag with amplitude. This increase may result in a reversal of the favorable effect of first-derivative lead as the lead factor r is indefinitely increased and suggests an optimum value for r, a conclusion that is supported by flight experience. This result may be traced to the counteracting effects of lead and amplitude introduced by the first-derivative lead. The first-derivative lead shifts the phase of the signal fed into the servo so that the signal leads θ by an amount $\tan^{-1} r\omega$ and increases the amplitude of the signal by the factor $\sqrt{1+r^2\omega^2}$. The change of lead with r may be stated symbolically as follows: If the dependence of ϵ_p on θ and ω without first-derivative lead is expressed by

$$\epsilon_p=f(\theta,\omega)$$

then the addition of first-derivative lead changes the lead to

$$\epsilon_p=f(\Theta,\omega)+\tan^{-1}r\omega \tag{4}$$

where

$$\Theta=\theta\sqrt{1+r^2\omega^2}$$

Differentiating equation (4) with respect to r gives

$$\frac{d\epsilon_p}{dr}=\frac{d\epsilon_p}{d\theta}\frac{d}{dr}\left(\theta\sqrt{1+r^2\omega^2}\right)+\frac{\omega}{1+r^2\omega^2}$$

$$=\frac{d\epsilon_p}{d\theta}\frac{\theta\omega^2 r}{\sqrt{1+r^2\omega^2}}+\frac{\omega}{1+r^2\omega^2} \tag{5}$$

If r is very small, the first term on the right-hand side of equation (5) is negligible, so that $\frac{d\epsilon_p}{dr}$ is positive. The addition, therefore, of a small amount of first-derivative lead results in a change in ϵ_p in the direction of more lead. A similar analysis of the effect of r on δ_p shows that $\frac{d\delta_p}{dr}=0$ when $r=0$. For small values of r, therefore, the effect of adding first-derivative lead is the same as for the linear servo: an improvement in dynamic stability and a reduction in the amplitude of hunting is to be expected. For large values of r, however, the lag due to increase in amplitude (first term) tends to counteract the direct lead (second term) and may cause more violent hunting. Some optimum value of r may therefore exist. A direct determination of its value may be made from the autopilot for various values of r, θ, and ω.

APPLICATION OF METHOD TO CONTROLS GEARED TO MOTION ABOUT TWO AXES

Sometimes the aileron or rudder is geared to both the angle of bank and the angle of yaw. Usually this gearing is accomplished by tilting the gyro in such a way that the rotation of the gyro is affected by bank and yaw according to the formula

$$\Phi=\phi\cos\tau+\psi\sin\tau$$

In order to apply the frequency-response method in this case it is necessary only to calculate the response in bank and yaw of the airplane to a sinusoidal control motion and to combine them according to the preceding formula to obtain Φ. This formula yields the δ_r and ϵ_r curves. The other pair of curves required—the δ_p and ϵ_p curves—are obtained in exactly the same way as before, namely, by oscillating the gyro about its sensitive axis. The critical control gearing so determined will be the critical ratio between δ and Φ.

CONCLUDING REMARKS

The frequency-response method of analysis is a useful graphical means of determining oscillation characteristics of an airplane equipped with an automatic pilot. If the servo is linear (ideal case), the critical control gearing beyond which increasing oscillations take place can be readily determined. If the servo has nonlinear characteristics (practical case), the existence, amplitude, and frequency of steady hunting oscillations can be determined approximately from the measured frequency response of the servo and the computed frequency response of the airplane.

In general, the use of first-derivative lead (a phase-shifting device) has a favorable effect on the dynamic stability. Large amounts of first-derivative lead or phase shift may be destabilizing due to the increase in the signal amplitude produced by this phase-shifting device.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *December 9, 1946*.

APPENDIX A

CALCULATION OF RESPONSE OF AN AIRPLANE TO SINUSOIDAL CONTROL MOTION

The response of the airplane in pitch to sinusoidal elevator motion will be used as an example. The basic equations are derived in reference 3 but with somewhat different notation and without the effects of control deflection, rate of change of angle of attack, and power. The equations of motion, for a given control deflection, are

$$\left.\begin{aligned}(x_u+\tau D)u+x_\alpha\alpha+x_\theta\theta=0\\ z_u u+(z_\alpha+\tau D)\alpha+(z_\theta-\tau D)\theta=-z_\delta\delta\\ C_{m_u}u+(C_{m_\alpha}+C_{m_{D\alpha}}D)\alpha+\left(C_{m_{D\theta}}D-\frac{\tau c}{2V}k_y^2D^2\right)\theta=-C_{m_\delta}\delta\end{aligned}\right\}\quad\text{(A1)}$$

where

$$x_u=C_D-\frac{1}{\rho VS}\frac{dT}{dV}$$

$$\tau=\frac{m}{\rho VS}$$

$$x_\alpha=-\frac{1}{2}(C_L-C_{D_\alpha})$$

$$x_\theta=\frac{C_L}{2}$$

$$z_u=C_L$$

$$z_\alpha=\frac{1}{2}(C_{L_\alpha}+C_D)$$

$$z_\theta=\frac{C_L\tan\Gamma_0}{2}$$

$$z_\delta=\frac{C_{L_\delta}}{2}$$

$$k_y^2=\frac{4I_y}{mc^2}$$

$$C_{m_u}=V\frac{\partial C_m}{\partial V}$$

$$C_{m_\alpha}=\frac{\partial C_m}{\partial\alpha}$$

. . . .

and

Γ_0 trim angle of climb, deg
T thrust, lb
I_y pitching moment of inertia, slug-ft²
m mass of airplane, slugs
c wing chord, ft
S wing area, sq ft
$u=\Delta V/V$
V velocity, ft/sec
ΔV change in velocity, ft/sec
C_L lift coefficient
α angle of attack, radians
C_m pitching-moment coefficient
C_D drag coefficient
ρ air density, slugs/cu ft

Solving equations (A1) for $\frac{\theta}{\delta}$ by the method of determinants gives

$$\frac{\theta}{\delta}=\frac{\begin{vmatrix}x_u+\tau D & x_\alpha & 0\\ z_u & z_\alpha+\tau D & -z_\delta\\ C_{m_u} & C_{m_\alpha}+C_{m_{D\alpha}}D & -C_{m_\delta}\end{vmatrix}}{\begin{vmatrix}x_u+\tau D & x_\alpha & x_\theta\\ z_u & z_\alpha+\tau D & z_\theta-\tau D\\ C_{m_u} & C_{m_\alpha}+C_{m_{D\alpha}}D & C_{m_{D\theta}}D-\frac{\tau c}{2V}k_y^2D^2\end{vmatrix}}$$

Expanding these determinants by the usual methods gives an expression for $\frac{\theta}{\delta}$ consisting of the ratio of two polynomials in D, which may be written as

$$\frac{\theta}{\delta}=\frac{a_1D^2+b_1D+c_1}{a_2D^4+b_2D^3+c_2D^2+d_2D+e_2}$$

The phase and amplitude of the response of θ to the motion $\delta=\sin\omega t$ is obtained by substituting $i\omega$ for D in the above expression for $\frac{\theta}{\delta}$. This substitution gives a complex number, say, $A+iB$. The angle of lead of θ ahead of δ is $\tan^{-1}\frac{B}{A}$ and the amplitude of θ is $\sqrt{A^2+B^2}$; that is

$$\epsilon_r=-\tan^{-1}\frac{B}{A}$$

$$R=\sqrt{A^2+B^2}$$

and

$$\theta=R\sin(\omega t-\epsilon_r)$$

$$\delta_r=\frac{1}{R}$$

APPENDIX B

DETERMINATION OF STABLE SIDE OF STABILITY BOUNDARY OBTAINED BY FREQUENCY-RESPONSE METHOD

It is shown in the body of this report that a critical value of the control gearing exists if at some frequency ω the values of ϵ_r and ϵ_p are equal and that the value of the critical control gearing is equal to δ_r/δ_{p_1} at that frequency. It will now be shown that the stable side of the stability boundary may be determined by the relative slopes of the ϵ_r and ϵ_p curves at the point where $\epsilon_r=\epsilon_p$.

As shown in reference 2, the response of an airplane to any disturbance may be computed from the "response function." The response to a sinusoidal disturbance is obtained by substituting $i\omega$ for the variable D in the response function. The response functions for the airplane and autopilot may be combined to form the stability equation of the airplane plus autopilot as follows:

Let $\theta=f_1(D)\delta$ be the airplane response function and $\delta=kf_2(D)\theta$ be the autopilot response function where k is the control gearing. Combining the two gives

$$\delta=kf_2(D)f_1(D)\delta$$

or

$$1-kf_2(D)f_1(D)=0$$

By substituting $f(D)=f_1(D)f_2(D)$ this equation may be written as

$$1-kf(D)=0 \tag{B1}$$

This is the "stability equation," the roots of which for the variable D determine the damping and frequency of the motion. The frequency responses are obtained from the response functions by use of the following:

$$f_1(i\omega)=Re^{-i\epsilon_r}=\frac{1}{\delta_r}\,e^{-i\epsilon_r}$$

$$f_2(i\omega)=\delta_{p_1}e^{i\epsilon_p}$$

Then a dynamic-stability boundary exists if $1-kf(i\omega)=0$ for some value of ω, say ω_{cr}. It follows from the definition of f_1 and f_2 that this dynamic-stability boundary occurs when

$$\epsilon_r-\epsilon_p=0$$

and

$$k=\frac{\delta_r}{\delta_{p_1}}=k_{cr}$$

at some value of $\omega=\omega_{cr}$. Under these conditions, equation (B1) has a pair of roots $D=\pm i\omega_{cr}$ when k has the value given.

In order to find the stable side of the boundary, it is necessary to find the sign of the real part of $\frac{dD}{dk}$ at the boundary. From equation (B1)

$$\frac{dD}{dk}=-\frac{f(D)}{kf'(D)}$$

This equation must be evaluated when $D=i\omega_{cr}$. At this value of ω, $f(i\omega_{cr})=\frac{1}{k_{cr}}$. In general,

$$f(i\omega)=\left(\frac{\delta_{p_1}}{\delta_r}\right)e^{i(\epsilon_p-\epsilon_r)}$$

Then

$$f'(D)=\frac{df(i\omega)}{d(i\omega)}=-i\,\frac{df(i\omega)}{d\omega}$$

therefore

$$\left[\frac{df(i\omega)}{d(i\omega)}\right]_{\omega=\omega_{cr}}=\frac{1}{k_{cr}}\left[\frac{d(\epsilon_p-\epsilon_r)}{d\omega}\right]_{\omega=\omega_{cr}}-i\left[\frac{d\left(\frac{\delta_{p_1}}{\delta_r}\right)}{d\omega}\right]_{\omega=\omega_{cr}}$$

and

$$\left(\frac{dD}{dk}\right)_{\omega=\omega_{cr}}=-\frac{1}{k_{cr}^2\left\{\frac{1}{k_{cr}}\left[\frac{d(\epsilon_p-\epsilon_r)}{d\omega}\right]_{\omega=\omega_{cr}}-i\left[\frac{d\left(\frac{\delta_{p_1}}{\delta_r}\right)}{d\omega}\right]_{\omega=\omega_{cr}}\right\}}$$

If $\left[\frac{d(\epsilon_p-\epsilon_r)}{d\omega}\right]_{\omega=\omega_{cr}}$ is negative, the real part of $\frac{dD}{dk}$ is positive. As k increases above the critical value, therefore, the roots of the stability equation indicate instability; that is, if

$$\left[\frac{d(\epsilon_p-\epsilon_r)}{d\omega}\right]_{\omega=\omega_{cr}}<0$$

then the stable region is located where k is less than k_{cr} and vice versa.

REFERENCES

1. Nyquist, H.: Regeneration Theory. Bell Syst. Tech. Jour., vol. XI, no. 1, Jan. 1932.
2. Jones, Robert T.: A Method for Studying the Hunting Oscillations of an Airplane with a Simple Type of Automatic Control. NACA Rep. No. 801, 1944.
3. Zimmerman, Charles H.: An Analysis of Longitudinal Stability in Power-Off Flight with Charts for Use in Design. NACA Rep. No. 521, 1935.